On The Hermite Based-Second Kind Genocchi Polynomials

Burak Kurt and Yilmaz Simsek

Department of Mathematics, Faculty of Science University of AkdenizTR-07058 Antalya, Turkey burakkurt@akdeniz.edu.tr and ysimsek@akdeniz.edu.tr

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Abstract

The aim of this paper is to study generating function of the Hermite-Kampė de Fėriet based second kind Genocchi polynomials. We also give some identities related to these polynomials.

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1. Introduction, Definitions and Notations

The Bernoulli numbers the Bernoulli polynomials, the Euler numbers, the Euler polynomials and Genocchi polynomials and numbers are used many areas of sciences. These numbers and polynomials are also used in calculus of finite differences, in numerical analysis, in analytical number theory.

Recently many authors have studied on these numbers and polynomials. They also defined many different generating functions of these polynomials and numbers.

Due to Bretti and Ricci [1], the Hermite-Kamp´e de F´eriet (or Gould-Hopper) polynomials have been used in order to construct addition formulas for different classes of generalized Gegenbauer polynomials. These polynomials (two-variable Hermite polynomials) are defined by means of the following generating functions:

$$e^{xt+yt^{j}}=\sum_{n=0}^{\infty}H_{n}^{(j)}\left(x,y\right)\frac{t^{n}}{n!}. \tag{1.1}$$

From (1.1), one can easily see that

$$H_{n}^{(j)}\left(x,y\right)=n!\sum_{r=0}^{\left[\frac{n}{j}\right]}\frac{x^{n-jr}y^{r}}{r!\left(n-jr\right)!},$$

where $j \geq 2$ is an integer cf. [1]. In [1], the case $j = 1$ is not considered, since the corresponding $2D$ polynomials are simply expressed by the Newton binomial formula.

The polynomials $H_n^{(j)}(x, y)$ are the solution of the generalized heat equation:

$$\begin{aligned} \frac{\partial}{\partial x} F(x,y) &= \frac{\partial^j}{\partial x^j} F(x,y), \\ F(x,0) &= x^n. \end{aligned}$$

The classical Genocchi numbers G_n and the classical Genocchi numbers $G_n(x)$ are usually defined by means of the following generating function,

$$g(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \ |t| < \pi \tag{1.2}$$

and

$$g(t,x) = g(t)e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \ |t| < \pi. \tag{1.3}$$

From (1.2) and (1.3), one can easily see that

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

The second kind Genocchi polynomials of higher order are defined by means of the following generating function:

$$\mathfrak{g}(t,x) = \left(\frac{2t}{e^t + e^{-t}} \right)^a e^{tx} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(a)}(x) \frac{t^n}{n!}, \ |t| < \frac{\pi}{2}. \tag{1.4}$$

Note that $\mathcal{G}_n^{(1)}(x) = \mathcal{G}_n(x)$ denotes the second kind Genocchi polynomials cf. [15]. $\mathcal{G}_n(0) = \mathcal{G}_n$ which denote so-called the second kind Genocchi numbers cf. [15].

By using (1.4), one can easily see that

$$\mathcal{G}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \mathcal{G}_k.$$

By (1.3) and (1.4), we have

$$\begin{aligned} \mathcal{G}_n(x) &= 2^{n-1} G_n(\frac{x+1}{2}) \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k, j, n-k-j} 2^{k-1} x^{n-k-j} G_k, \end{aligned}$$

where

$$\binom{n}{a, b, c} = \frac{n!}{a!b!c!}, a + b + c = n.$$

The second kind Euler polynomials of higher order are defined by means of the following generating function:

$$\left(\frac{2}{e^{t}+e^{-t}}\right)^{a} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(a)}(x) \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2} \tag{1.5}$$

Observe that $\mathcal{E}_n^{(1)}(x) = \mathcal{E}_n(x)$ denotes the second kind Euler polynomials cf. ([12], [16]).

Relation between $\mathcal{E}_n(x)$ and $\mathcal{G}_n(x)$ is given by

$$\mathcal{G}_n(x) = n\mathcal{E}_{n-1}(x).$$

Lemma 1. *The second kind Genocchi polynomials of α order is satisfied the following relations*

$$G_n^{(\alpha+\beta)}(x+x_1) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha)}(x) G_{n-k}^{(\beta)}(x_1). \tag{1.6}$$

Also

$$\sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)} G_{n-k}^{(\alpha)}(x+x_1) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)}(x)\, G_{n-k}^{(\alpha)}(x_1). \tag{1.7}$$

Proof. From (1.4),

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(\alpha+\beta)}(x+x_1)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n^{(\beta)}(x_1)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} G_k^{(\alpha)}(x) G_{n-k}^{(\beta)}(x_1)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients, we prove Lemma 1. □

2. Hermite-Kampė de Fėriet based second kind Genocchi polynomials

The aim of this section is to define the Hermite-Kampė de Fėriet (or Gould-Hopper) based second kind Genocchi polynomials. Some poroperties of these polynomials are also given.

The Hermite-Kampė de Fėriet based second kind Genocchi polynomials of higher order are defined by means of the following generating function:

$$F_{H,a}(t,x,y;j) = \left(\frac{2t}{e^{t}+e^{-t}}\right)^{a} e^{xt+yt^{j}} = \sum_{n=0}^{\infty} \left({}_{H}\mathcal{G}_n^{(j,a)}(x,y)\right)\frac{t^n}{n!}. \tag{2.1}$$

From (2.1), one finds that

$${}_{H}\mathcal{E}_{n-1}^{(j,a)}(x,y) = \frac{{}_{H}\mathcal{G}_n^{(j,a)}(x,y)}{n},$$

where ${}_{H}\mathcal{E}_{n-1}^{(j,a)}(x,y)$ denotes the Hermite-Kampė de Fėriet based second kind Euler polynomials of higher order.

Now we compute the derivative of ((2.1) with respect to x to derive a derivative formula for the Hermite-Kampė de Fėriet based second kind Genocchi polynomials of higher order:

$$\frac{\partial}{\partial x}\left({}_{H}\mathcal{G}_n^{(j,a)}(x,y)\right) = n\left({}_{H}\mathcal{G}_{n-1}^{(j,a)}(x,y)\right).$$

If we substitute $j=2$ and $a=1$ into (2.1), and using (1.4), we have

$$\sum_{n=0}^{\infty}\left({}_{H}\mathcal{G}_{n}^{(2,1)}\left(x,y\right)\right)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\mathcal{G}_{n}(x)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}\frac{y^{n}t^{2n}}{n!}. \tag{2.2}$$

After some elementary calculations in the above, we obtain

$$\sum_{n=0}^{\infty}\left({}_{H}\mathcal{G}_{n}^{(2)}\left(x,y\right)\right)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left(n!\sum_{l=0}^{\left[\frac{n}{2}\right]}\frac{y^{l}\mathcal{G}_{n-2l}(x)}{l!(n-2l)!}\right)\frac{t^{n}}{n!}.$$

By comparing coefficients of t^{n} in the both sides of the above, we arrive at the result of the theorem:

Theorem 1.

$$ {}_{H}\mathcal{G}_{n}^{(2,1)}\left(x,y\right)=n!\sum_{l=0}^{\left[\frac{n}{2}\right]}\frac{y^{l}\mathcal{G}_{n-2l}(x)}{l!(n-2l)!}.$$

Theorem 2.

$$ {}_{H}\mathcal{G}_{n}^{(j,a+b)}\left(x_{1}+x_{2},y_{1}+y_{2}\right)=\sum_{k=0}^{n}\binom{n}{k}\left({}_{H}\mathcal{G}_{n}^{(j,a)}\left(x_{1},y_{1}\right)\right)\left({}_{H}\mathcal{G}_{n-k}^{(j,b)}\left(x_{2},y_{2}\right)\right). \tag{2.3}$$

Proof. By (2.1), we define the following functional equation

$$F_{H,a+b}(t,x_{1}+x_{2},y_{1}+y_{2};j)=F_{H,a}(t,x_{1},y_{1};j)F_{H,b}(t,x_{2},y_{2};j)$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty}\left({}_{H}\mathcal{G}_{n}^{(j,a+b)}\left(x_{1}+x_{2},y_{1}+y_{2}\right)\right)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left({}_{H}\mathcal{G}_{n}^{(j,a)}\left(x_{1},y_{1}\right)\right)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}\left({}_{H}\mathcal{G}_{n}^{(j,b)}\left(x_{2},y_{2}\right)\right)\frac{t^{n}}{n!}$$

By using Cauchy product in the above equation, than comparing the coefficients of $\frac{t^{n}}{n!}$ on both sides, we arrive at the desired result. □

Relation between Hermite-based second kind Genocchi polynomials of higher order, the second kind Genocchi polynomials of higher order and two-variable generalized Hermite polynomials is given by the next theorem.

Theorem 3.

$$ {}_{H}\mathcal{G}_{n}^{(2,a)}(x,y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{G}_{n-l}^{(a)}H_{l}^{(2)}\left(x,y\right).$$

Proof. If we substitute $j=2$ and $a=1$ into (2.1), we easily arive at the desired result. □

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